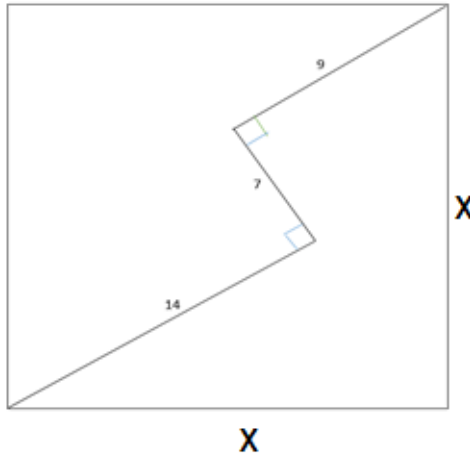
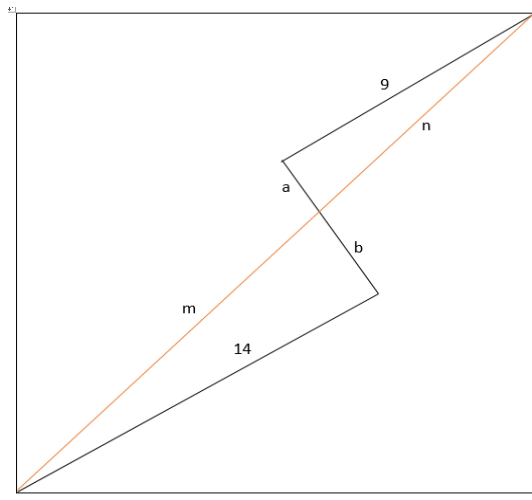


1. Find the side length "X" of the square in the following case:



**Solution:**

Draw the diagonal and name the segments as shown:



The two right triangles obtained are similar, therefore:

$$\frac{a}{b} = \frac{9}{14} \quad (1)$$

Given:  $a + b = 7$  (2)

Solve this system of equation (1, 2) by substituting:  $b = \frac{14}{9}a$  in (2)

$$a\left(1 + \frac{14}{9}\right) = \frac{23}{9}a = 7 \quad \text{so: } a = \frac{63}{23} \quad \text{and: } b = \frac{98}{23}$$

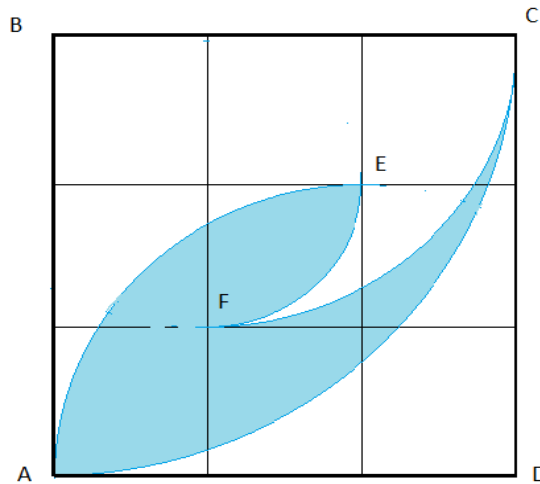
In the same triangles also we know the length of diagonal is:  $m + n = \sqrt{2} X$ , in which m and n can be calculated using Pythagorean theorem, separately:

$$m^2 = 14^2 + \left(\frac{98}{23}\right)^2 = \frac{14^2 \cdot 23^2 + 14^2 \cdot 7^2}{23^2}, \quad \text{So: } m = \frac{14}{23} \sqrt{23^2 + 7^2} = \frac{14}{23} \sqrt{578} = \frac{14}{23} \cdot 17\sqrt{2}$$

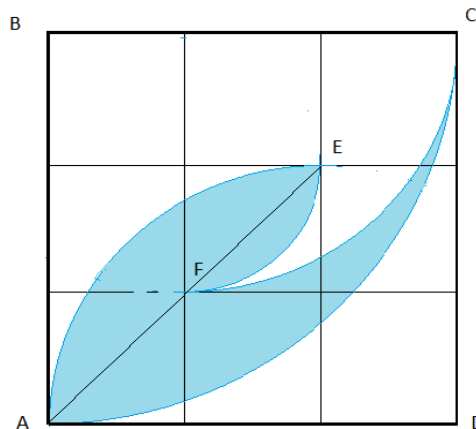
$$n^2 = 9^2 + \left(\frac{63}{23}\right)^2 = \frac{9^2 \cdot 23^2 + 9^2 \cdot 7^2}{23^2}, \quad \text{So: } n = \frac{9}{23} \sqrt{23^2 + 7^2} = \frac{9}{23} \sqrt{578} = \frac{9}{23} \cdot 17\sqrt{2}$$

Therefore:  $m + n = \left(\frac{14}{23} + \frac{9}{23}\right) \cdot 17\sqrt{2} = 17\sqrt{2} = \sqrt{2} X$  Then:  $X = 17$  ✓

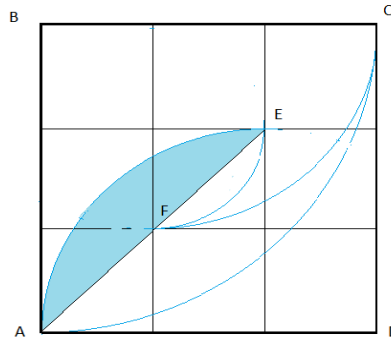
2. Calculate the shaded area in the 3x3 diagram bellow.



**Solution:** The line AE divides the shaded area in 3 separate areas:

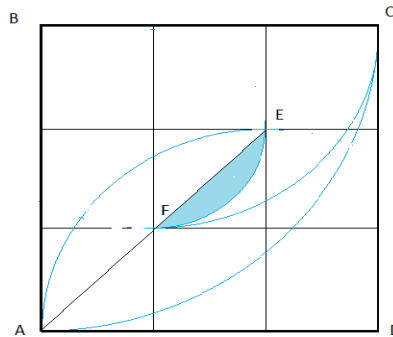


- a. A quarter of circle with a radius of 2 units minus the triangle in its sector:



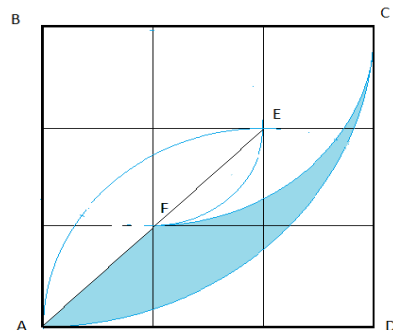
$$A = \frac{\pi(2^2)}{4} - \frac{1}{2}(2 \times 2) = \pi - 2$$

- b. A quarter of circle with a radius of 1 unit minus the triangle in its sector:



$$B = \frac{\pi(1^2)}{4} - \frac{1}{2}(1 \times 1) = \frac{\pi}{4} - \frac{1}{2}$$

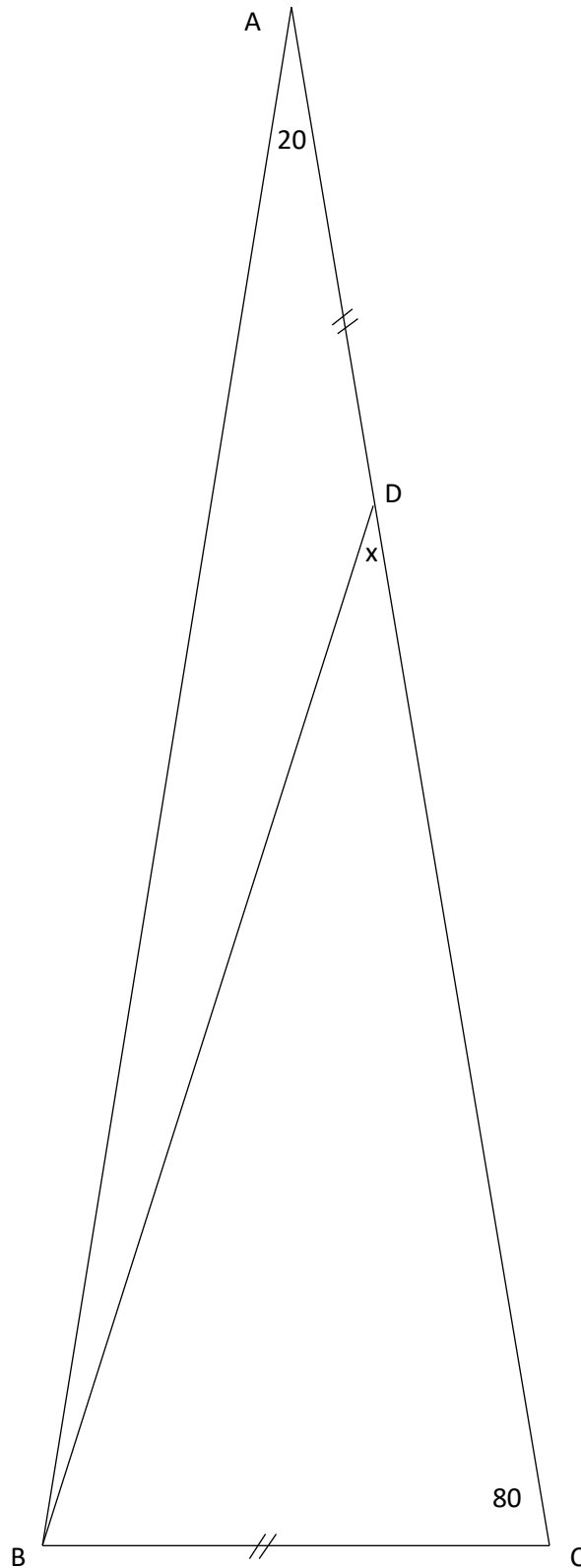
- c. The area enclosed by 2 quarter of circles with R = 3 and 2, and the line AF:



$$C = \frac{\pi(3^2)}{4} - \frac{\pi(2^2)}{4} - \frac{3+2}{2} \times 1 = \frac{9\pi}{4} - \pi - \frac{5}{2} = \frac{5\pi}{4} - \frac{5}{2}$$

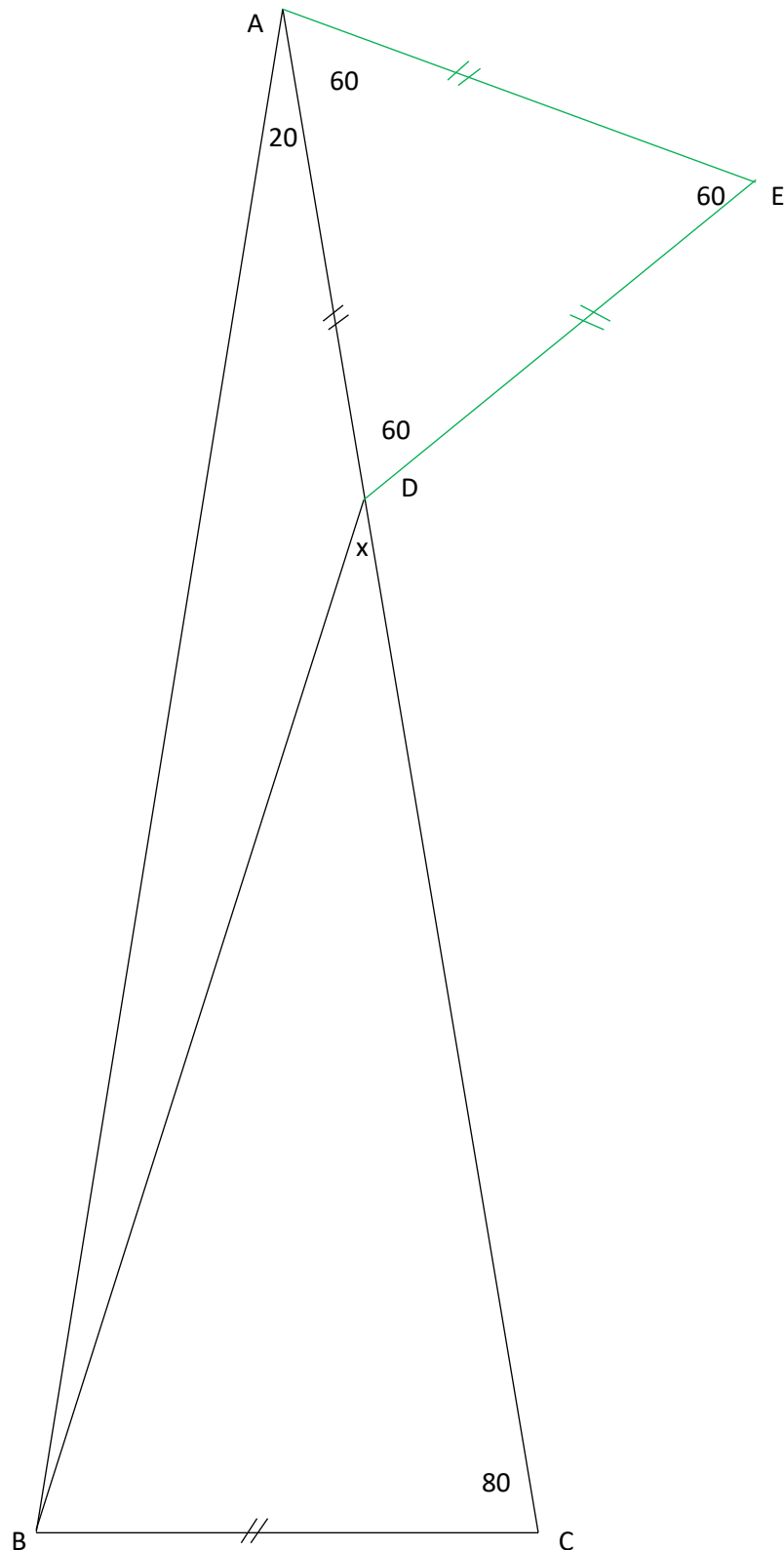
So the total shaded area is:  $A + B + C = \frac{5\pi - 10}{2} = \frac{5}{2} (\pi - 2)$  ✓

3. In the triangle ABC, if  $A = 20^\circ$ ,  $C = 80^\circ$ , and  $AD = BC$ . Then find the angle  $x$ ?

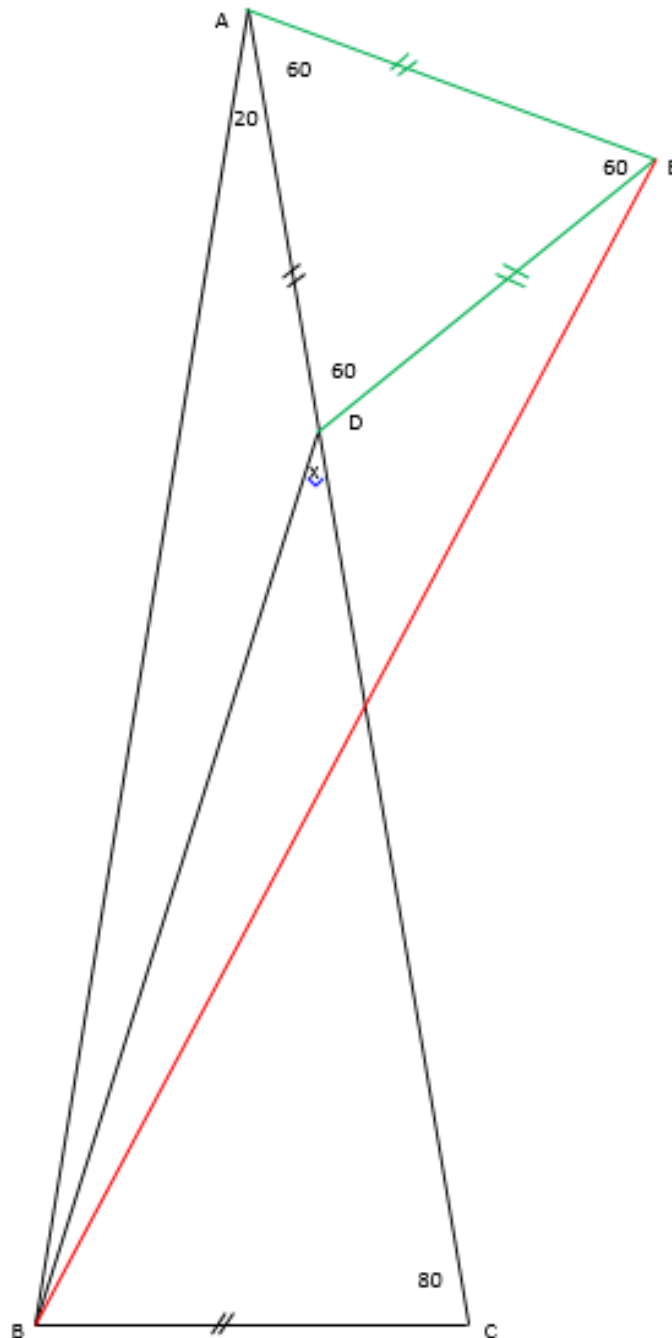


**Solution:** Triangle ABC is Isosceles, because the angle ABC is also  $80^\circ$  then  $AB = AC$

Draw the equilateral tringle AED



Connect E to B. Triangles ABC and ABE are congruent, SAS, sharing side AB, AE = BC, and the angle:  $\angle ABC = \angle BAE = 80^\circ$



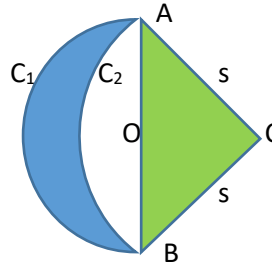
Therefore angle AEB is also  $80^\circ$ . The triangle AEB is Isosceles and  $AB = EB$  so the angle  $DEB = 20^\circ$ .

Conclusion: triangles ABD and EDB are congruent, Therefore angles  $\angle ABD = \angle EBD = 10^\circ$  then the angle CBD is  $70^\circ$

So the angle x in the triangle ABD is  $30^\circ$  ✓

4. In the following diagram, ABC is right, isosceles triangle with sides S.  $C_1$  is a semi-circle with diameter AB centred at O midpoint of AB and  $C_2$  is a quarter of circle with radius S and centre C.

Prove the area of the triangle is equal to the blue area enclosed by two arcs.



**Solution:**

The area of the triangle ABC is:  $\frac{S^2}{2}$  ✓

The hypotenuse of the triangle  $AB = S\sqrt{2}$ , then the radius of  $C_1$  is:  $\frac{S\sqrt{2}}{2}$ .

The area of the semi-circle  $C_1$  is:  $\frac{1}{2}\pi\left(\frac{S\sqrt{2}}{2}\right)^2 = \frac{\pi S^2}{4}$

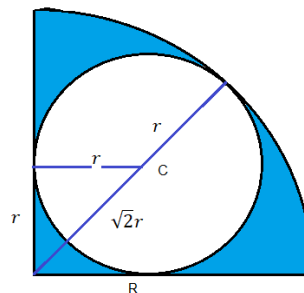
The area of the quarter circle  $ACBC_2$  is:  $\frac{\pi S^2}{4}$ , so the area of white spot is:  $\frac{\pi S^2}{4} - \frac{S^2}{2}$

Therefore the area of the blue zone is the semi-circle area minus the white part:

$$\frac{\pi S^2}{4} - \left(\frac{\pi S^2}{4} - \frac{S^2}{2}\right) = \frac{S^2}{2} \quad \checkmark$$

Which is equal to the area of the triangle.

5. In the following diagram, a circle with radius  $r$  is inscribed in a quarter of a larger circle with radius  $R$ . Find the ratio of the shaded to the inner circle area.



**Solution:**

$$R = r + \sqrt{2}r = r(1 + \sqrt{2}) \quad \text{Shown diagonally}$$

The total area of the quarter sector is:  $A = \frac{\pi}{4} R^2 = \frac{\pi}{4} r^2 (1 + \sqrt{2})^2 = \frac{\pi}{4} r^2 (3 + 2\sqrt{2}) = (3\frac{\pi}{4} + \frac{\pi}{2}\sqrt{2})r^2$

The area of the inner circle is:  $B = \pi r^2$

So the shaded area is  $A - B = (3\frac{\pi}{4} + \frac{\pi}{2}\sqrt{2})r^2 - \pi r^2 = (-\frac{\pi}{4} + \frac{\pi}{2}\sqrt{2})r^2 = \frac{\pi}{4}(2\sqrt{2} - 1)r^2$

Therefore the ratio is:  $\frac{2\sqrt{2}-1}{4}$  ✓

6. Find the smallest positive three-digit integer  $N$ , which has a remainder of 2 when divided by 6, a remainder of 5 when divided by 9 and a remainder of 7 when divided by 11.

**Solution:**

According to the divisions rule, results can be represented as:

$$\frac{N}{6} = a + \frac{2}{6}, \quad \frac{N}{9} = b + \frac{5}{9}, \quad \frac{N}{11} = c + \frac{7}{11}$$

In which  $a$ ,  $b$  and  $c$  are also integers and respectively divisible by 6, 9 and 11 with:

The LCM of the denominator will have factors of: 2, 3, 3, and 11. Therefore the LCM is:  $2.3.3.11 = 198$

Refer to the division rules again, it is known if we subtract the remainder from the number  $N$  or we add the difference between the divisor and the remainder to  $N$ , the result of division will be an integer.

Note: 
$$\frac{D (\text{dividend})}{d (\text{divisor})} - Q (\text{quotient}) = \frac{R (\text{remainder})}{d}$$

The difference of the divisor and remainder in each case is 4, using division property that means:

$$\frac{N}{6} = \frac{198 - 4}{6} = a, \quad \text{remainder} = 2$$

$$\frac{N}{9} = \frac{198 - 4}{9} = b, \quad \text{remainder} = 5$$

$$\frac{N}{11} = \frac{198 - 4}{11} = c, \quad \text{remainder} = 7$$

Therefore:  $N = 194$  with:  $a = 34$   $R=2$ ,  $b = 21$   $R=5$  and  $c = 17$   $R=7$  ✓



7. a, b, c are 3 consecutive terms of an arithmetic sequence, Prove:
- (a + b), (a + c) and (b + c) are also consecutive terms of an arithmetic sequence.
  - $\frac{1}{\sqrt{b}+\sqrt{c}}$ ,  $\frac{1}{\sqrt{a}+\sqrt{c}}$  and  $\frac{1}{\sqrt{a}+\sqrt{b}}$  are also consecutive terms of an arithmetic sequence.

**Solution:**

- a. Let's "a" be the first term, then "b = a+d" and "c = a+2d" will be the next terms.

Therefore: a+b = 2a+d, a+c = 2a+2d and b+c = 2a+3d. These terms make a new arithmetic sequence with the same common difference d. ✓

$$\text{b. } \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a+d}+\sqrt{a+2d}} \cdot \frac{\sqrt{a+d}-\sqrt{a+2d}}{\sqrt{a+d}-\sqrt{a+2d}} = \frac{\sqrt{a+d}-\sqrt{a+2d}}{a+d-a-2d} = \frac{\sqrt{a+d}-\sqrt{a+2d}}{-d} = \frac{\sqrt{a+2d}-\sqrt{a+d}}{d} \quad (1)$$

$$\frac{1}{\sqrt{a}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{a+2d}} \cdot \frac{\sqrt{a}-\sqrt{a+2d}}{\sqrt{a}-\sqrt{a+2d}} = \frac{\sqrt{a}-\sqrt{a+2d}}{a-a-2d} = \frac{\sqrt{a}-\sqrt{a+2d}}{-2d} = \frac{\sqrt{a+2d}-\sqrt{a}}{2d} \quad (2)$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{a+d}} \cdot \frac{\sqrt{a}-\sqrt{a+d}}{\sqrt{a}-\sqrt{a+d}} = \frac{\sqrt{a}-\sqrt{a+d}}{a-a-d} = \frac{\sqrt{a}-\sqrt{a+d}}{-d} = \frac{\sqrt{a+d}-\sqrt{a}}{d} \quad (3)$$

To prove these 3 terms represent an arithmetic sequence, calculate (3 - 2) and (2 - 1):

$$\frac{\sqrt{a+d}-\sqrt{a}}{d} - \frac{\sqrt{a+2d}-\sqrt{a}}{2d} = \frac{2\sqrt{a+d}-2\sqrt{a}-\sqrt{a+2d}+\sqrt{a}}{2d} = \frac{2\sqrt{a+d}-\sqrt{a+2d}-\sqrt{a}}{2d} = D$$

$$\frac{\sqrt{a+2d}-\sqrt{a}}{2d} - \frac{\sqrt{a+2d}-\sqrt{a+d}}{d} = \frac{\sqrt{a+2d}-\sqrt{a}-2\sqrt{a+2d}+2\sqrt{a+d}}{2d} = \frac{2\sqrt{a+d}-\sqrt{a+2d}-\sqrt{a}}{2d} = D$$

D is the common difference. ✓

8. Three men—A, B, and C—crossed paths walking through woods on a cold night. They decided to light a fire to rest by, and set out to gather some firewood. A came back with 5 logs of wood, B brought 3 logs, but C came back empty-handed. C requested that they let him rest by the fire and promised to pay them some money in the morning. In the morning C paid them \$8. How should A and B split the money fairly?

- A \$7; B \$1
- A \$6; B \$2
- A \$5; B \$3

- d. A \$4; B \$4
- e. None of these

**Solution:**

All three men are equally benefited by the fire from the 8 logs of wood. Each man used  $\frac{8}{3}$  logs of wood through the night. Therefore,

A contributed  $5 - \frac{8}{3} = \frac{7}{3}$  logs of wood.

B contributed  $3 - \frac{8}{3} = \frac{1}{3}$  log of wood, and C zero.

Therefore they must share \$8 in proportion to  $\frac{7}{3}:\frac{1}{3}$  and  $\frac{0}{3}$  so A \$7 and B \$1. ✓

9. Multiple identities in one question:

If A, B and C are the 3 angles of a triangle, prove:  $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \cdot \sin B \cdot \sin C$  (1)

**Solution:**

Recall to the following identities:  $\sin 2A = 2\sin A \cdot \cos A$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B, \quad \text{and} \quad \cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

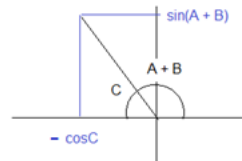
a. We prove:  $\sin 2A + \sin 2B = 2 \sin(A+B) \cdot \cos(A-B)$

$$\text{RHS} = 2 (\sin A \cdot \cos B + \cos A \cdot \sin B)(\cos A \cdot \cos B + \sin A \cdot \sin B)$$

$$= 2(\sin A \cdot \cos A \cdot \cos^2 B + \sin B \cdot \cos B \cdot \sin^2 B + \sin B \cdot \cos B \cdot \cos^2 A + \sin A \cdot \cos A \cdot \sin^2 B)$$

$$= 2\sin A \cdot \cos A (\cos^2 B + \sin^2 B) + 2\sin B \cdot \cos B (\sin^2 B + \cos^2 A) = \sin 2A + \sin 2B \quad \checkmark$$

b. Given:  $A + B + C = \pi$ , we can see:



i.  $\sin(A+B) = \sin C \quad \checkmark$

ii.  $\cos(A+B) = -\cos C \quad \checkmark$

c.  $\cos(A-B) - \cos(A+B) = \cos A \cdot \cos B + \sin A \cdot \sin B - (\cos A \cdot \cos B - \sin A \cdot \sin B) = 2\sin A \cdot \sin B \quad \checkmark$

d. Prove of identity (1):

$$\begin{aligned} \text{According to a. LHS: } \sin 2A + \sin 2B + \sin 2C &= 2 \sin(A+B) \cdot \cos(A-B) + \sin 2C \\ &= 2 \sin(A+B) \cdot \cos(A-B) + 2 \sin C \cdot \cos C \\ &= 2 \sin C \cdot \cos(A-B) + 2 \sin C \cdot \cos C \quad (\text{according to b.i}) \\ &= 2 \sin C \cdot (\cos(A-B) + \cos C) = 2 \sin C \cdot (\cos(A-B) - \cos(A+B)) \quad (\text{according to b.ii}) \end{aligned}$$

Then according to c.  $= 2 \sin C \cdot (2 \sin A \cdot \sin B) = 4 \sin A \cdot \sin B \cdot \sin C$

Therefore:  $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \cdot \sin B \cdot \sin C$  ✓

10. in the following sequence:  $\left(1 - \frac{1}{2}\right), \left(2 - \frac{2}{3}\right), \left(3 - \frac{3}{4}\right), \dots, \left(15 - \frac{15}{16}\right)$

- a. write the general term in simplest form
- b. find the product of all terms

**Solution:**

a.  $U_n = n - \frac{n}{n+1}$  ✓

b. After simplifying each term:

$$P = \left(\frac{1}{2}\right) \cdot \left(\frac{4}{3}\right) \cdot \left(\frac{9}{4}\right) \cdot \left(\frac{16}{5}\right) \cdot \left(\frac{25}{6}\right) \cdot \left(\frac{36}{7}\right) \dots \dots \dots \left(\frac{225}{16}\right)$$

As we can see the numerator is the product of consecutive integers square ( $n^2$ ) from 1 to 15 and the denominator is the product of consecutive integers ( $n+1$ ) from 2 to 16.

After crossing out common factors from numerator with denominator we get:

$$P = 1.2.3.4.5 \dots \dots \dots \frac{15}{16} = \frac{15!}{16} \checkmark$$